AUTOMORPHIC COHOMOLOGY ON HOMOGENEOUS COMPLEX MANIFOLDS*

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§1. *Introduction*

Automorphic forms on the upper half plane have been under investigation since the late nineteenth century (see the classic text by Fricke-Klein [*7*], for instance). In several variables automorphic forms were studied by Blumenthal, Hilbert, and Picard near the turn of the century and more recently by Siegel [*14*] on the Siegel upper half space. The study of automorphic forms on general Lie groups was introduced by Harish-Chandra [*11*] and has been a subject of considerable interest (cf. Borel [*4*] and Murakami [*12*]). In particular, automorphic forms on a bounded symmetric domain can be considered as holomorphic sections of homogeneous vector bundles induced by finite-dimensional representations of a compact isotropy group, the classical case corresponding to powers of the canonical bundle (see Borel [*4*]). Automorphic forms on a domain D are defined with respect to a discrete transformation Γ group acting on D, and the quotients of any two forms (of the same weight) are meromorphic functions on the quotient space D/Γ. This quotient space may have singularities as an analytic space depending on the nature of Γ, and thus the classical study of automorphic forms was a means of studying the function field on certain classes of analytic spaces. This is the somewhat tenuous relation of the present note to the theme of this conference on "singularities."

Suppose D is a bounded Hermitian symmetric domain of the form $D = G/K$ where G is a semisimple real Lie group and K is a maximal compact subgroup. Let ρ be a finite-dimensional unitary representation of K, and let E_ρ be the associated homogeneous vector bundle over D (cf. §*2*). If Γ is a discrete subgroup of G which acts properly discontinuously on D, then the Γ-invariant sections of $H^0(D, E_\rho)$, denoted by $H^0_\Gamma(D, E_\rho)$, are the automorphic forms on D (with additional growth restrictions at the cusps in the case $\dim_{\mathbb{C}} D = 1$). Generalizations of such Hermitian bounded

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symmetric domains arise in algebraic geometry as the classifying spaces for Hodge structures on compact algebraic (more generally Kähler) manifolds (Griffiths [9], [10]). In this situation such domains D are homogeneous complex manifolds of the form G/V where G is a semisimple Lie group and V is a compact (but not necessarily maximal compact) subgroup of G. If E is a homogeneous vector bundle over D associated to a finite-dimensional representation ρ of V, then Schmid has shown that

$$H^0(D, E_\rho) = 0$$

for most representations ρ in the case where V is not maximal compact (so that D is *not* an Hermitian bounded symmetric domain, see Schmid [13], Griffiths-Schmid [10]). Thus there are *no* automorphic forms on these generalizations of the upper half plane or Siegel's upper half space (which are the classifying spaces for the Hodge structures on Riemann surfaces). In general, Schmid has shown that $H^q(D, E_\rho)$ is an infinite-dimensional Frechet space which should be the analogue of $H^0(D, E_\rho)$ in the classical case of bounded symmetric domains. Here $q = \frac{1}{2}\dim_{\mathbf{R}} K/V$ where K is the maximal compact subgroup of G containing V (which turns out to be the same as the complex dimension of the maximal-dimensional compact complex submanifold of D). The Γ-invariant cohomology classes $H^q_\Gamma(D, E_\rho)$ are the natural candidates for "automorphic forms" on D. These are the *automorphic cohomology classes* of Griffiths (Griffiths [9], [10], Griffiths-Schmid [10]).

At present very little is known about this particular version of "automorphic forms," except in the case where D/Γ is compact (see Griffiths-Schmid [10]). In particular, if Γ is the natural generalization of Siegel's modular group, then it is unknown whether $H^q_\Gamma(D, E_\rho)$ is finite-dimensional. In this note we announce some new results concerning the geometry of these more general homogeneous complex manifolds $D = G/V$. It is possible that the techniques developed here will be useful in attacking the above problem of finite dimensionality of the automorphic cohomology groups. Specifically, we show that for a sufficiently nonsingular representation ρ, the automorphic cohomology classes on D can be represented as the Γ-invariant sections of a holomorphic vector bundle over a Stein manifold M naturally and equivariantly associated to D. Here M is essentially the space of compact q-dimensional subvarieties of D. A consequence of this result is the convergence of the Poincaré series

$$\theta(\phi) = \sum_{\gamma \in \Gamma} \gamma^* \phi$$

where $\phi \in H^q(D, E_\rho)$ is an absolutely integrable cohomology class (with respect to a suitable metric). Such a Poincaré series is an automorphic cohomology class, and the classical study of automorphic forms was very dependent upon understanding such series.

Many of the results announced here were conjectured by Griffiths [*9*], [*10*]. Details will appear elsewhere at a later date.

§2. *Homogeneous Complex Manifolds*

Let $G_{\mathbf{C}}$ be a connected complex semisimple Lie group and let $P \subset G_{\mathbf{C}}$ be a parabolic subgroup, i.e., the coset space $X = G_{\mathbf{C}}/P$ is a compact homogeneous complex manifold (which is known to be projective algebraic). Let $G \subset G_{\mathbf{C}}$ be a real form of $G_{\mathbf{C}}$ such that $V = G \cap P$ is a compact subgroup of $G_{\mathbf{C}}$. Then the G-orbit of the coset $eP \in G_{\mathbf{C}}/P$ is an open subset D of the complex manifold X. The manifold $D = G/V$ is a homogeneous complex manifold, and it is the geometry of such manifolds in which we are principally interested. An example is given by the real Lorentz group $G = SO(2h, k)$, where $V = U(h) \times SO(k)$, which arises as the classifying space for periods of holomorphic 2-forms on a given projective algebraic manifold of complex dimension $\geqq 2$ (cf. Griffiths [*8*]). In this case $G = SO(2h + k, \mathbf{C})$, and P is given explicitly in Griffiths-Schmid [*10*, p. 262] (see this paper and also Wolf [*17*] for further background on this class of homogeneous complex manifolds). The subgroup $P \subset G_{\mathbf{C}}$ is parabolic if and only if P contains a Borel subgroup B (i.e., a maximal solvable subgroup), and there is a natural fibration

$$G_{\mathbf{C}}/B \to G_{\mathbf{C}}/P$$

whose fibers are compact complex submanifolds isomorphic to P/B. The manifolds $G_{\mathbf{C}}/B$ are flag manifolds and are somewhat easier to study than the case of an arbitrary parabolic subgroup (cf. Bott [*5*], Schmid [*13*], Griffiths-Schmid [*10*]).

Suppose now that K is a maximal compact subgroup of G which contains V. Then it is known that the image of the injection

$$K/V \to G/V = D$$

is a complex submanifold Y of the complex manifold D (cf. Borel [*3*], Griffiths-Schmid [*10*]). Let Y_g denote the translate of the submanifold $Y \subset X$ by $g \in G_{\mathbf{C}}$, and let $\mathcal{Y}_X$ denote the disjoint union of the complex submanifolds of X, each of which is of the form Y_g for some $g \in G_{\mathbf{C}}$. The space $\mathcal{Y}_X$ admits a complex structure such that the mapping

$$\mathcal{Y}_X \xrightarrow{\tau} X$$

induced by the inclusion mapping on fibers is holomorphic. Moreover, there is a parametrization mapping

$$\mathcal{Y}_X \xrightarrow{\pi} S$$

where S is an algebraic homogeneous submanifold of $\mathbf{C}^N$ for some N. Namely, the mapping π is a proper surjective holomorphic mapping of maximal rank such that $\pi^{-1}(s)$, for $s \in S$, is one of the fibers in the original disjoint union $\mathcal{Y}_X$, i.e., $\pi^{-1}(s) = Y_g$ for some $g \in G_{\mathbf{C}}$ (cf. Griffiths [*8*], Wells [*15*], and Windham [*16*]). Let $\mathcal{Y}_D = \tau^{-1}(D)$ be the union of those translates of Y which are contained in D, and let $M = \pi(\mathcal{Y}_D) \subset S$ be the parameter space for these translates.

Theorem 1. *M is a Stein manifold.*

The proof of this result uses the construction of an appropriate exhaustion function for D due to Schmid [*13*] (cf. Griffiths-Schmid [*10*]), the solution of the Levi problem for spaces of analytic cycles due to Andreotti-Norguet [*1*], [*2*], and an exhaustion principle for Stein manifolds due to Docquier-Grauert [*6*]. A special case of this result was proven explicitly in Wells [*15*] for the case $G = SO(2h, 1)$, $V = U(h)$.

§3. *Homogeneous Vector Bundles*

Let $D = G/V$ be a homogeneous complex manifold of the type considered in the previous section. Suppose that

$$\rho : V \to GL(E)$$

is a representation of V on the finite-dimensional complex vector space E. Consider the equivalence relation in the trivial vector bundle $G \times E \to G$:

$$\{(g, e) \in G \times E : (g, e) = (gv, \rho(v)e), \quad \text{for} \quad v \in V\}.$$

This equivalence relation induces in a natural manner a vector bundle E_ρ on the coset space $G/V = \{gV\}_{g \in G}$, which is called the *homogeneous vector bundle on D induced by the representation* ρ (cf. Bott [*5*]). The adjective "homogeneous" denotes the fact that the action of G on the homogeneous space G/V lifts to an action on the vector bundle E defined over G/V.

Suppose now that we are given a homogeneous vector bundle $E_\rho \to D$ as above, and suppose that ρ is *sufficiently nonsingular*, i.e., the highest weight of the representation ρ is sufficiently bounded away from the boundary of the Weyl chamber of positive roots associated with the Lie algebra of $G_{\mathbf{C}}$ and a maximal torus $T \subset V$ (note that the choice of Weyl chamber is dictated by the choice of the complex structure on D, which in our setting is given a priori; see Schmid [13], Griffiths-Schmid [10]). This condition was used by Schmid in his study of the infinite-dimensional representations of G (Schmid [13]), and can be prescribed precisely in terms of the root structure of the Lie algebra in a given situation. In case $P = B$ is a Borel subgroup of $G_{\mathbf{C}}$, then the condition above is satisfied by a sufficiently high power of the canonical bundle. Classically this would correspond to automorphic forms of high weight. In the case of a homogeneous line bundle, the highest weight is the differential of the (one-dimensional) representation and thus defines an element in the dual space to the Lie algebra.

Let τ^*E be the pullback of E_ρ to $\mathcal{Y}_D$, and let

$$\tilde{E}_\rho = \pi_*^q \tau^* E$$

be the qth direct image of τ^*E on M, where $q = \dim_{\mathbf{C}} Y$ (we are thinking of the direct image sheaves as vector bundles, since in this case the direct image sheaves are locally free). Then $\tilde{E}_\rho$ is a vector bundle on M which has fibers isomorphic to $H^q(Y, E_{\rho|Y})$. Moreover, the action of G on D induces an action on $H^r(D, E_\rho)$ and on $H^r(M, \tilde{E}_\rho)$, for all $r \geqq 0$. Schmid has shown that if ρ is sufficiently nonsingular, then

$$H^r(D, E_\rho) = 0, \quad r \neq q,$$

$$\dim H^q(D, E_\rho) = \infty,$$

and moreover, $H^q(D, E_\rho)$ has the natural structure of a Frechet space inherited from the natural topology on spaces of C^∞ differential forms and Dolbeault's isomorphism. The difficult part is to show that the range of $\bar{\partial}$ is closed so that the quotient space in the Dolbeault group is Hausdorff (see Schmid [13]).

Theorem 2. *Suppose the representation ρ is sufficiently nonsingular. Then there is a G-equivariant topological injection*

$$\xi \colon H^q(D, E_\rho) \to H^0(M, \tilde{E}_\rho). \tag{1}$$

By "topological" in the above theorem we mean that the algebraic injection is a closed subspace of the Frechet space $H^0(M, \tilde{E}_\rho)$. The proof of this theorem breaks down into two parts. First we see that the Leray spectral sequence relating cohomology on $\mathcal{Y}_D$ to cohomology on M degenerates, giving us an isomorphism

$$H^q(\mathscr{Y}_D, \tau^* E_\rho) = H^0(M, \tilde{E}_\rho),$$

and thus giving a natural Frechet space structure to $H^q(\mathscr{Y}_D, \tau^* E_\rho)$. This follows from the fact that M is Stein (Theorem 1) and Cartan's Theorem B on M (this doesn't use the homogeneous nature of the vector bundles). A different proof can be obtained as a consequence of the fact that the remaining direct image sheaves $\pi^j_*(\tau^* E_\rho) = 0$ if $j \neq q$, which follows from the vanishing theorem of Bott [*5*] along with the local triviality of the fibration $\mathscr{Y}_D \to M$. The natural mapping

$$\tau^*: H^q(D, E_\rho) \to H^q(\mathscr{Y}_D, \tau^* E_\rho)$$

is then shown to be a topological injection by using a "power series expansion" of the cohomology classes about the fiber Y in D due to Schmid [*13*]. Here the homogeneous nature of the vector bundles is used quite strongly. It seems likely that an alternative proof valid in greater generality can be constructed by using an appropriate generalization of Grauert's direct image theorem to the case of a real-analytic fibering of compact complex manifolds (which structure D has in a natural way, namely the fibering $G/V \to G/K$).

§4. *Automorphic Cohomology*

Using the above results we obtain a representation for automorphic cohomology and the convergence of certain Poincaré series. If Γ is a discrete subgroup of G which acts properly discontinuously on D by left translation, then Γ induces an action on $H^q(D, E_\rho)$ and on $H^0(D, \tilde{E}_\rho)$ by Theorem 2. Let $H^q_\Gamma(D, E_\rho)$ be the Γ-invariant subspace of $H^q(D, E_\rho)$ and let $H^0_\Gamma(M, \tilde{E}_\rho)$ be the Γ-invariant subspace of $H^0(M, \tilde{E}_\rho)$. As an immediate consequence of Theorem 2 we have the following result.

Theorem 3. *The mapping ξ in (1) induces a topological injection*

$$\xi: H^q_\Gamma(D, E_\rho) \to H^0_\Gamma(M, \tilde{E}_\rho).$$

Thus Theorem 3 tells us that automorphic cohomology can be interpreted as invariant sections of a vector bundle. This had been conjectured by Griffiths [*8*]. In particular we note that we do not lose any information about the automorphic cohomology as we pass to sections; however, we do not expect every Γ-invariant section of $\tilde{E}_\rho$ to come from an automorphic cohomology class, as simple examples show that this is highly unlikely in this particular type of geometric situation.

Let $E_\rho \to D$ be as above and let E_ρ and the tangent bundle to D be equipped with G-invariant metrics. Using these metrics it is easy to make precise what it means for a $(0, q)$ differential form with coefficients in E_ρ to be in L^1 (absolutely integrable). We then say that a cohomology class in $H^q(D, E_\rho)$ is in L^1 if it has a Dolbeault representative which is in L^1. Let $H_1^q(D, E_\rho)$ be the subspace of L^1 cohomology classes in $H^q(D, E_\rho)$ (cf. Griffiths [8]).

As noted by Griffiths ([8], p. 619), the power series arguments of Schmid in [13] show that $H_1^q(D, E_\rho)$ is nonempty (and infinite dimensional) provided that ρ is sufficiently nonsingular. We then have the following theorem conjectured by Griffiths in [8], where ρ is as above.

Lemma 4. *Let* $\phi \in H_1^q(D, E_\rho)$, *and let*

$$\theta(\phi) = \sum_{\gamma \in \Gamma} \gamma^* \phi; \tag{2}$$

then (2) *converges to an automorphic cohomology class in* $H_\Gamma^q(D, E_\rho)$.

This theorem was partially proved in Griffiths [8] where he showed that the associated Poincaré series on M

$$\theta(\xi(\phi)) = \sum_{\gamma \in \Gamma} \gamma^*(\xi(\phi)) \tag{3}$$

converges in $H^0(M, \tilde{E}_\rho)$, using basic convergence arguments on this series of sections of a vector bundle stemming from the more classical theory. But it now follows easily from Theorem 2 and the convergence of (3) that the series in (2) converges. We strongly use the fact that the image of ξ in the mapping (1) is closed.

If $H^q(D, E_\rho)$ is finite dimensional as in the classical case, then, as was pointed out by Poincaré in his original paper on Poincaré series (which he called "theta-Fuchsian series"), it is necessary that almost all of the series of the form (2) vanish identically.

Remark. In case the manifold D studied above fibers holomorphically over an Hermitian bounded symmetric domain (i.e., $G/V \to G/K$ is a holomorphic mapping), then the above results are much easier as it is unnecessary to consider the deformation space $\mathcal{Y}_D$ introduced in §2. This is the case when $D = G/V$ admits a Kähler metric invariant under G and can be determined by the structure of the Lie algebras involved (Borel [3]). However, the classifying spaces for Hodge structures (Griffiths [8], [9]), for instance the example given in §2, do *not* have this property.

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